22087210

International Baccalaureate®
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MATHEMATICS
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PAPER 3 – SERIES AND DIFFERENTIAL EQUATIONS

Monday 19 May 2008 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

3 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 6]*

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ is convergent or divergent.

2. *[Maximum mark: 9]*

(a) Using l'Hopital's Rule, show that $\lim_{x\to\infty} x\mathrm{e}^{-x} = 0$. *[2 marks]*

(b) Determine $\int_0^a x\mathrm{e}^{-x}\,\mathrm{d}x$. *[5 marks]*

(c) Show that the integral $\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x$ is convergent and find its value. *[2 marks]*

3. *[Maximum mark: 13]*

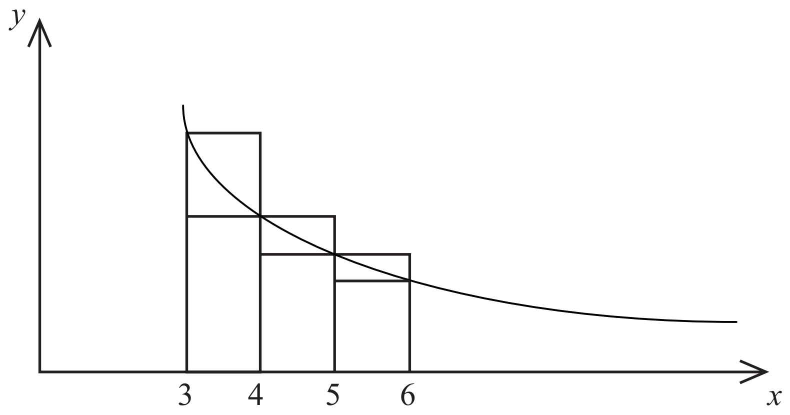
Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \frac{x^3}{x^2+1}.$$

(a) Find an integrating factor for this differential equation. *[5 marks]*

(b) Solve the differential equation given that $y = 1$ when $x = 1$, giving your answer in the form $y = f(x)$. *[8 marks]*

4. *[Maximum mark: 15]*



The diagram shows part of the graph of $y = \frac{1}{x^3}$ together with line segments parallel to the coordinate axes.

(a) Using the diagram, show that

$$\frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \ldots < \int_3^\infty \frac{1}{x^3}\,\mathrm{d}x < \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \ldots.$$ *[3 marks]*

(b) **Hence** find upper and lower bounds for $\sum_{n=1}^{\infty} \frac{1}{n^3}$. *[12 marks]*

5. *[Maximum mark: 17]*

The function f is defined by

$$f(x) = \ln\left(\frac{1}{1-x}\right).$$

(a) Write down the value of the constant term in the Maclaurin series for $f(x)$. *[1 mark]*

(b) Find the first three derivatives of $f(x)$ and hence show that the Maclaurin series for $f(x)$ up to and including the x^3 term is $x + \frac{x^2}{2} + \frac{x^3}{3}$. *[6 marks]*

(c) Use this series to find an approximate value for $\ln 2$. *[3 marks]*

(d) Use the Lagrange form of the remainder to find an upper bound for the error in this approximation. *[5 marks]*

(e) How good is this upper bound as an estimate for the actual error? *[2 marks]*